

IV. *A general Method for exhibiting the Value of an Algebraic Expression involving several Radical Quantities in an Infinite Series: Wherein Sir Isaac Newton's Theorem for involving a Binomial, with another of the same Author, relating to the Roots of Equations, are demonstrated. By T. Simpson F. R. S.*

Read Jan. 10.
1750-1.

AMONG all the great improvements, which the art of computation hath in these last ages received, the method of series may be justly one of the most considerable; since not only the doctrine of chances and annuities, with some other branches of the mathematics, depend almost intirely thereon, but even the business of fluents, of such extensive use, would, without its aid and concurrence, be quite at a stand in a multitude of cases, as is well known to mathematicians.

It is for this reason, that the celebrated binomial theorem, for converting radical quantities into series's, is ranked, by many, among the principal discoveries of its illustrious author; seeing, by means thereof, a vast number of fluents are found, that would otherwise be impracticable: nor is there any case, however complex, to which it may not be extended.

It is true, when two or more compound radical quantities are involved together, the operation, by having two or more series's to multiply into one
another,

another, becomes very troublesome and laborious; and, what is worse, the Law of continuation, whereby a part of the labour might be avoided, is exceedingly hard, if not impossible, this way to be discovered. In the following paper something is attempted towards obviating the said inconveniencies; but whether the success has been answerable, I shall not take upon me to determine.

PROBLEM I.

To find a series exhibiting the value of $\sqrt[m]{1 + \frac{x}{a}}$

$\times \sqrt[n]{1 + \frac{x}{b}} \times \sqrt[p]{1 + \frac{x}{c}} \times \sqrt[q]{1 + \frac{x}{d}}$ &c. in simple terms; x being indeterminate, and $a, b, c, d, m, n, p, \&c.$ any given numbers, whole or broken, positive or negative.

Put $u = \sqrt[m]{1 + \frac{x}{a}}$, $w = \sqrt[n]{1 + \frac{x}{b}}$, $y = \sqrt[p]{1 + \frac{x}{c}}$, $z = \sqrt[q]{1 + \frac{x}{d}}$ &c.

Also let $\Delta = u w y z, \&c.$ (= the quantity proposed)

Then, in fluxions $\dot{\Delta} = \dot{u} w y z, \&c. + u \dot{w} y z, \&c. + u w \dot{y} z, \&c. + u w y \dot{z}, \&c.$ Which equation, divided by the preceding one, gives $\frac{\dot{\Delta}}{\Delta} = \frac{\dot{u}}{u} + \frac{\dot{w}}{w} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} \&c.$

But,

But, since $u = 1 + \sqrt[m]{\frac{x}{a}}$, we have $\dot{u} = m\dot{x} \times 1 + \sqrt[m]{\frac{x}{a}}^{m-1}$;

and therefore $\frac{\dot{u}}{u} = \frac{m\dot{x}}{a} \times \sqrt[m]{\frac{x}{a}}^{-1} = \frac{m\dot{x}}{a} \times$

$1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} \&c. \&c.$ by Division.

And in the same manner it appears, that $\frac{\dot{w}}{w} = \frac{n\dot{x}}{b} \times 1 - \frac{x}{b} + \frac{x^2}{b^2} \&c. \&c.$

Hence, our equation, by substituting these values, becomes

$$\frac{\dot{\Delta}}{\Delta} = \dot{x} \times \left\{ \begin{array}{l} \frac{m}{a} - \frac{mx}{a^2} + \frac{mx^2}{a^3} - \frac{mx^3}{a^4} \&c. \\ \frac{n}{b} - \frac{nx}{b^2} + \frac{nx^2}{b^3} - \frac{nx^3}{b^4} \&c. \\ \frac{p}{c} - \frac{px}{c^2} + \frac{px^2}{c^3} - \frac{px^3}{c^4} \&c. \\ \&c. \quad \&c. \quad \&c. \quad \&c. \end{array} \right\}$$

$$\text{Put } P = \frac{m}{a} + \frac{n}{b} + \frac{p}{c} + \frac{q}{d} \&c.$$

$$Q = \frac{m}{a^2} + \frac{n}{b^2} + \frac{p}{c^2} + \frac{q}{d^2} \&c.$$

$$R = \frac{m}{a^3} + \frac{n}{b^3} + \frac{p}{c^3} + \frac{q}{d^3} \&c.$$

Then it will be

$$\frac{\dot{\Delta}}{\Delta} = \dot{x} \times P - Qx + Rx^2 - Sx^3 + Tx^4 - Vx^5 \&c.$$

Assume

Assume $\Delta = A + Bx + Cx^2 + Dx^3 + Ex^4$, &c.
let this value, with that of Δ , be substituted in the
last equation: from whence, by comparing the ho-
mologous terms, there will come out

$$B = PA$$

$$C = \frac{PB - QA}{2}$$

$$D = \frac{PC - QB - RA}{3}$$

$$E = \frac{PD - QC + RB - SA}{4}$$

$$F = \frac{PE - QD + RC - SB + TA}{5}$$

$$G = \frac{PF - QE + RD - SC + TB - VA}{6}$$

&c.

Where the law of continuation is manifest, and
where it is also evident, that the value of (A) the
first term of the required series, must be an unit;
because, when $x=0$. then the given expression be-

comes $1 \times 1 \times 1 = 1$. $Q.E.I.$

COROL. I.

If a be taken $= 1$, and n, p, q , &c. each $= 0$;
then will $P=m$, $Q=m$, $R=m$, &c. And there-
fore

$$A = 1$$

$$B = m$$

$$2C = mB - mA$$

$3D$

$$3D = mC - mB + mA = mC - 2C$$

$$4E = mD - mC + mB - mA = mD - 3D$$

∴ c.

$$\text{Consequently } C = \frac{m \cdot \overline{m-1}}{2}, D = \frac{C \times \overline{m-2}}{3} = \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{2 \cdot 3}, E = \frac{D \times \overline{m-3}}{4} = \frac{m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \overline{m-3}}{2 \cdot 3 \cdot 4}$$

∴ c.

$$\text{Hence, in this case, } 1 + mx + \frac{m \cdot \overline{m-1}}{2} x^2 + \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{2 \cdot 3} x^3 \text{ ∴ c. } (= A + Bx + Cx^2 \text{ ∴ c.}) =$$

$\overline{1+x}^m$: which series is the same with that given by Sir Isaac Newton.

COROL. 2.

If a be taken $= \frac{1}{a}$, $\beta = \frac{1}{b}$, $\gamma = \frac{1}{c}$, ∴ c. and $z = \frac{1}{x}$ then will the proposed expression be transformed to

$$\overline{1 + \frac{\alpha}{z}}^m \times \overline{1 + \frac{\beta}{z}}^n \times \overline{1 + \frac{\gamma}{z}}^p \times \overline{1 + \frac{\delta}{z}}^q \text{ ∴ c.}$$

$$\text{Also } P = m\alpha + n\beta + p\gamma + \text{∴ c.}$$

$$Q = m\alpha^2 + n\beta^2 + p\gamma^2 + \text{∴ c.}$$

$$R = m\alpha^3 + n\beta^3 + p\gamma^3 + \text{∴ c.}$$

∴ c.

$$\text{And consequently } \overline{1 + \frac{\alpha}{z}}^m \times \overline{1 + \frac{\beta}{z}}^n \times \overline{1 + \frac{\gamma}{z}}^p \times \overline{1 + \frac{\delta}{z}}^q \text{ ∴ c.}$$

$\phi c. = A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} \phi c.$ where $A=1$, $B=PA$, $C=\frac{PB-QA}{2} \phi c.$ as before. Which equation or theorem answers in case of a descending series.

COROL. 3.

Hence, if each of the quantities $m, n, p, \phi c.$ be taken equal to unity, and their number be denoted by v ; then will $1 + \frac{\alpha}{z} \times 1 + \frac{\beta}{z} \times 1 + \frac{\gamma}{z} \times 1 + \frac{\delta}{z} \phi c. = A + \frac{B}{z} + \frac{C}{z^2} + \frac{D}{z^3} \phi c.$ Which equation, multiplied by z^v , gives $\overline{z+\alpha} \times \overline{z+\beta} \times \overline{z+\gamma} \times \overline{z+\delta} \phi c. = Az^v + Bz^{v-1} + Cz^{v-2} + Dz^{v-3} \phi c.$

Whence it appears, that $\overline{z-\alpha} \times \overline{z-\beta} \times \overline{z-\gamma} \times \overline{z-\delta} \phi c.$ is $= Az^v - Bz^{v-1} + Cz^{v-2} - Dz^{v-3} \phi c.$ Where $A=1$, $B=PA$, $C=\frac{PB-QA}{2}$, $D=\frac{PC-QB+RA}{3}$, $\phi c.$ (as before); P being in this case = sum of all the quantities $\alpha, \beta, \gamma, \delta, \phi c.$ Q = the sum of all their squares; R = the sum of their cubes, $\phi c. \phi c.$

COROL. 4.

Since $\alpha, \beta, \gamma, \delta, \phi c.$ are the roots of the equation, $z^v - Bz^{v-1} + Cz^{v-2} - Dz^{v-3}, \phi c. = 0$; it follows,

follows, that, if $B, C, D, E, \&c.$ be given; the sum of those roots (P); the sum of their squares (Q), and the sum of their cubes (R) $\&c.$ will also be given from the foregoing equations: whence will be had

$$\begin{aligned} P &= B \\ Q &= +PB - 2C \\ R &= -PC + QB + 3D \\ S &= +PD - QC + RB - 4E \\ T &= -PE + QD - RC + SB + 5F \\ \&c. & \qquad \&c. \end{aligned}$$

where the law of continuation is obvious.

These values are the same with those given (without demonstration) by Sir Isaac Newton, in his Universal Arithmetic, for finding when some of the roots of an equation are impossible.

PROBLEM II.

To find a series expressing the value of $1 + \sqrt[m]{\frac{x}{a}}$
 $\times 1 + \sqrt[n]{\frac{x^2}{b}}$ $\times 1 + \sqrt[p]{\frac{x^3}{c}}$ $\times 1 + \sqrt[q]{\frac{x^4}{d}}$, $\&c.$

By putting $u = 1 + \sqrt[m]{\frac{x}{a}}$, $w = 1 + \sqrt[n]{\frac{x^2}{b}}$, $\&c.$; and proceeding as in the last problem; there will be had

$$\frac{\dot{u}}{u} = \frac{m\dot{x}}{a} \times 1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} \&c.$$

$$\frac{\dot{w}}{w} = \frac{2u\dot{x}}{b} \times 1 - \frac{x^2}{b} + \frac{x^4}{b^2} - \frac{x^6}{b^3} \&c.$$

$\&c. \qquad \&c.$

Whence,

Whence, making $P = \frac{m}{a}$, $Q = \frac{m}{a^2} - \frac{2n}{b}$, $R = \frac{m}{a^3} + \frac{3p}{c}$, $S = \frac{m}{a^4} + \frac{2n}{b^2} - \frac{4q}{d}$, $T = \frac{m}{a^5} + \frac{5r}{e}$, &c. and assuming $A + Bx + Cx^2 + Dx^3 + Ex^4$, &c. to express the series sought, the several values of A , B , C , D , &c. will be exhibited by the very equations brought out in the resolution of the preceding problem.

V. *A Letter from George Bayly M. D. of Chichester, to Henry Pemberton M. D. F. R. S. &c. of the Use of the Bark in the Small-Pox.*

Dear Sir,

Read Jan. 10. 1750. **T**HE case I lately mention'd to you in conversation, of which you desired a more particular account, is, as far as I have been able to recollect at this distance of time, as follows.

The patient, a gentlewoman of a fat corpulent habit, and healthy constitution, but 73 years of age, was, on the 6 day of December 1742, seiz'd with the common symptoms of a fever, attended with a sudden great loss of strength; so that, being carried to bed, she was not able to sit upright in it for the least space of time, without being held up by her assistant.